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Functional Classes with "Dominated" Mixed Derivative and the K-Functional*

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Classes of function W_p^r, $r=(r_1, ..., r_n)$, $1<p<\infty$, with "dominated" mixed derivative are considered. A new formula for the K-functional of the couple $(L_p; W_p^r)$ is proved. The functional spaces generated by the real method of interpolation are described. © 1995 Academic Press, Inc.

Let f be a function of n real variables that are 2π-periodic in each variable and such that

$$\int_{-\pi}^{\pi} f(x_1, ..., x_n)\, dx_j = 0, \qquad j=1, ..., n.$$

Denote by $f^{(\mathbf{r})}$ a mixed derivative of order $\mathbf{r}=(r_1, ..., r_n)$ in the Weil sense. If $f^{(\mathbf{r})} \in L_p(\mathbb{T}^n)$, $1<p<\infty$, we say that f belongs to the class $W_p^{\mathbf{r}}$. For convenience sake, consider the coordinates $r_1, ..., r_n$ as being in nondecreasing order of magnitude.

Let $\|f\|_{W_p^r} = \|f\|_{p,\mathbf{r}} \overset{\text{def}}{=} \|f^{(\mathbf{r})}\|_p$. Beginning with Babensko's paper [1] many papers have devoted to the investigation of these classes (see, for example, [2]).

In this paper we give a formula for the K-functional of the couple $(L_p; W_p^{\mathbf{r}})$ in the Ciesielski form and describe the interpolating space generated by the real method of interpolation.

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1. Let us remember the definition of the K-functional (see, for example, [3]). Peetre's K-functional is given by the formula

$$K(t,f;L_p;W_p^{\mathbf{r}})=\inf_{f=g+h}(\|g\|_p+t\,\|h\|_{p,\mathbf{r}}),$$

where the infimum is taken over all representations $f=g+h$.

Let

$$\rho(s)=\{\mathbf{k}\in\mathbb{Z}^n\colon 2^{s_j-1}\leqslant|k_j|<2^{s_j}, j=1,...,n\}$$

be sets of indices for every $s\in\mathbb{Z}_+^n$.

We denote the segments of the Fourier series of f by

$$\delta_{\mathbf{s}}(f;\mathbf{x})=\sum_{\mathbf{k}\in\rho(s)}\hat{f}(k)\,e^{i(k,x)}.$$

THEOREM 1. *Let $1<p<\infty$ and $2^N=[1/t]$. Then*

$$K(t^{r_1};f;L_p;W_p^{\mathbf{r}})\sim\left\|\frac{1}{2^{Nr_1}}\sum_{(\mathbf{s},\mathbf{r})\leqslant r_1N}\frac{\partial^{r_1+\cdots+r_n}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\delta_{\mathbf{s}}(f;\mathbf{x})+\sum_{(\mathbf{s},\mathbf{r})>r_1N}\delta_{\mathbf{s}}(f;\mathbf{x})\right\|_p.$$

Proof. It is based on the Marcinkievicz multiplier theorem and Bohr's and Bernstein's inequalities [2]. From the definition and the Marcinkievicz theorem we have

$$\begin{aligned}K(t^{r_1};f;L_p;W_p^{\mathbf{r}})&\leqslant\left\|\sum_{(\mathbf{s},\mathbf{r})>r_1N}\delta_{\mathbf{s}}(f;\mathbf{x})\right\|_p\\&\quad+\frac{1}{2^{Nr_1}}\left\|\sum_{(\mathbf{s},\mathbf{r})\leqslant r_1N}\frac{\partial^{r_1+\cdots+r_n}}{\partial^{r_1}_{x_1}\cdots\partial x_n^{r_n}}\delta_{\mathbf{s}}(f;\mathbf{x})\right\|_p\\&\leqslant C_p\left\|\frac{1}{2^{Nr_1}}\sum_{(\mathbf{s},\mathbf{r})\leqslant r_1N}\frac{\partial^{r_1+\cdots+r_n}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\delta_{\mathbf{s}}(f;\mathbf{x})\right.\\&\quad\left.+\sum_{(\mathbf{s},\mathbf{r})>r_1N}\delta_{\mathbf{s}}(f;\mathbf{x})\right\|_p.\end{aligned}$$

In order to prove the lower estimate it suffices to show the following two inequalities: The first is

$$\left\|\frac{1}{2^{Nr_1}}\sum_{(\mathbf{s},\mathbf{r})\leqslant r_1N}\frac{\partial^{r_1+\cdots+r_n}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}}\delta_{\mathbf{s}}(g;\mathbf{x})+\sum_{(\mathbf{s},\mathbf{r})\leqslant r_1N}\delta_{\mathbf{s}}(g;\mathbf{x})\right\|_p\leqslant C_p\,\|g\|_p$$

and it follows from Bernstein's inequality and the Marcinkievicz theorem.

Bohr's inequality and the Marcinkiewicz theorem yield the second inequality:

$$\left\| \frac{1}{2^{Nr_1}} \sum_{(\mathbf{s},\mathbf{r}) \leqslant r_1 N} \frac{\partial^{r_1+\cdots+r_n}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} \delta_{\mathbf{s}}(h;\mathbf{x}) + \sum_{(\mathbf{s},\mathbf{r}) > r_1 N} \delta_{\mathbf{s}}(h;\mathbf{x}) \right\|_p \leqslant \frac{C(p,\mathbf{r})}{2^{Nr_1}} \|h\|_{p,\mathbf{r}}.$$

The theorem is proved.

2. Let us consider the following linear polynomial method of summability

$$R_N(f;\mathbf{x}) = \sum_{(\mathbf{s},\mathbf{r}) \leqslant r_1 N} \left[\delta_{\mathbf{s}}(f;\mathbf{x}) - \frac{1}{2^{Nr_1}} \frac{\partial^{r_1+\cdots+r_n}}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} \delta_{\mathbf{s}}(f;\mathbf{x}) \right],$$

the so-called "step-hyperbolic" Riesz means [2]. Thus the theorem can be rewritten in the form

$$K(t^{r_1}; f; L_p; W_p^{\mathbf{r}}) \simeq \|f - R_N(f)\|_p,$$

and the class of interpolating spaces generated by the real method is defined by the seminorm [3]

$$\|f\|_{\theta,q} = \begin{cases} \sum_{k=0}^{\infty} ((2^{k\theta r_1} \|f - R_k(f)\|_p)^q)^{1/q}, & 0 < \theta \leqslant 1;\ 1 \leqslant q < \infty \\ \sup_k 2^{k\theta r_1} \|f - R_k(f)\|_p, & q = \infty. \end{cases}$$

Using the Marcinkiewicz multiplier theorem we can describe these spaces by means of best approximation by trigonometric polynomials with harmonics from the hyperbolic cross.

Let $E_k(f; L_p)$ be the error in the best approximation to f by trigonometric polynomials with harmonics from the set

$$\bigcup_{(\mathbf{s},\mathbf{r}) \leqslant r_1 N} \rho_{\mathbf{s}}(f).$$

STATEMENT. *For* $0 < \theta < 1$ *and* $1 \leqslant q < \infty$,

$$\|f\|_{\theta,q} \simeq \left(\sum_{k=0}^{\infty} (2^{k\theta r_1} E_k(f; L_p))^q \right)^{1/q}.$$

Proof. The lower estimate is obvious. When proving the upper estimate we use the Marcinkievicz multiplier theorem. It yields

$$\sum_{k=0}^{\infty} (2^{k\theta r_1} \|f - R_k(f)\|_p)^q$$
$$\leqslant \sum_{k=0}^{\infty} 2^{k\theta r_1 q} \left(\sum_{j=0}^{\infty} \left(2^{(j-k) r_1} E_j(f; L_p) + \sum_{j=k+1}^{\infty} E_j(f; L_p) \right)^q \right.$$

The conclusion now is a consequence of Hardy's well-known inequality. The statement is proved.

Remark. In [4] an estimate of the best approximation by "hyperbolic cross" via the mixed modulus of continuity is given.

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